

Supersymmetry and non-Abelian T-duality in type II supergravity

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Abstract

We study the effect of T-duality on supersymmetry in the context of type II supergravity. For both U(1) Abelian and SU(2) non-Abelian T-duality, we demonstrate that the supersymmetry variations after T-duality are related to the variations before T-duality through the Kosmann spinorial Lie derivative, which vanishes when the Killing spinors are independent of the T-duality directions. As a byproduct of our analysis, we present closed expressions for SU(2) T-duality in a class of spacetimes with diagonal Bianchi IX symmetry and comment on specific examples of T-dual geometries, including a novel AdS₃ geometry with large $\mathcal{N} = (4, 4)$ superconformal symmetry.

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1 Introduction

(Abelian) T-duality has been very successfully used over the years as a solution generating technique in string theory. Being a genuine symmetry of string theory, both in its perturbative and low energy expansions, the newly generated solution is guaranteed to produce a consistent string theory background, which should exhibit the same basic properties as the original one. Still, puzzling situations arise when the dual theory is explored in its low energy limit, such as the emergence of singularities and the explicit breaking of spacetime supersymmetries [1, 2, 3, 4]. It is well-known that those symmetries of the original theory, non-commuting with the ones that generate the duality, and supersymmetry is no exception to it, are only realized non-locally in the dual [5], and are thus typically non-manifest in the low energy expansion.

The non-Abelian cousin of Abelian T-duality was first explored in the 90's at the level of the string sigma model [6]. Although very similar at this level to its Abelian counterpart, this transformation did not reach the status of a string theory symmetry for two main reasons. First, the derivation in [6] only works at tree level in string perturbation theory, and second, it is only known to respect conformal symmetry to first order in α' . Yet, starting with the derivation of the transformation rules of the RR potentials by Sfetsos and Thompson in [7], this transformation has been very successfully used in recent years to generate new supergravity backgrounds with putative CFT duals, which in many cases share striking similarities with CFTs derived in the literature in completely different set-ups. Interesting examples are the relation found in [7] between an $SU(2)$ dual of $AdS_5 \times S^5$ and the $\mathcal{N} = 2$

Gaiotto-Maldacena geometries [8], its $\mathcal{N} = 1$ counterpart, worked out in [9, 10], which relates an $SU(2)$ dual of Klebanov-Witten [11] with the $\mathcal{N} = 1$ CFTs in [12, 13], the connection between an $SU(2)$ dual [14, 15] of the $AdS_6 \times S^4$ Brandhuber and Oz background [16] and the 5d fixed-point theories classified in [17], or the explicit realization of the $\mathcal{N} = 2$ 3d CFT that must be dual to the new AdS_4 background constructed in [18]. These examples show that, in contrast with its Abelian counterpart, non-Abelian T-duality may produce AdS/CFT pairs with very different properties from the original ones.

What is clear is that non-Abelian T-duality is a powerful solution generating technique on par with Ehlers or Geroch transformations in pure gravity. Although it has been explored largely on a case-by-case basis, for example [7, 10, 14, 18, 19], it has been proved that the simplest form of non-Abelian T-duality, namely a left or right-acting $SU(2)$ transformation, is a symmetry of the equations of motion (EOMs) of type II supergravity for spacetimes with $SO(4)$ isometry [20]. In this paper, we extend the analysis to a much larger class of spacetimes with Bianchi IX symmetry and, eschewing the EOMs - partial results for a class of spacetimes with $SU(2)$ isometry can be found in [21] - show how non-Abelian T-duality transforms the supersymmetry conditions of type II supergravity ¹. Indeed, since supersymmetry implies the Einstein and dilaton equations under mild assumptions, by combining our results with existing integrability results [26, 27] ², it should be feasible to show that non-Abelian T-duality is a generic solution generating technique in type II supergravity. We leave this direction to future work.

A further motivation for this work is to derive, in an alternative way, the transformation for the Ramond-Ramond (RR) sector under a non-Abelian T-duality transformation. We recall that the transformation for the RR sector under non-Abelian T-duality, which was originally proposed as a transformation on the flux bispinor using the relative twist of left and right movers [7], can be derived via dimensional reduction [20, 21] and can be inferred from the Fourier-Mukai transformation [29]. Here we offer an alternative derivation using the gravitino supersymmetry variation, which largely parallels the Abelian calculation [30]. It would be interesting to recover similar results by studying vertex operators [31] or pure spinor formalism [32], approaches which have led to the desired result in the Abelian case.

The general treatment of supersymmetry in this work also raises the possibility that non-Abelian T-duality may be useful for identifying new supersymmetry-preserving integrable-deformations in the context of AdS/CFT. It is known that the application of a sequence of (Abelian) T-duality - shift - T-duality (TsT) transformations leads to integrable deformations of $AdS_5 \times S^5$ [33, 34]. Recently, it has been shown that non-Abelian T-duals arise as limits of integrable theories [35] (see also [36, 37]), further implying that non-Abelian T-duality preserves integrability. Some hint of this can be found in studies of semi-classical strings in T-dual geometries [38, 39], and it is expected that the methods of [40], recently applied to the related complex β -deformation of $\mathcal{N} = 4$ super-Yang-Mills [41], will not observe chaotic behaviour ³. It would be interesting to see if analogous TsT transformations can be found for non-Abelian T-duality.

In this paper we take the first steps towards exploring the issue of spacetime supersym-

¹This covers specific examples where supersymmetry has been studied in the context of G -structures [22, 23, 24, 25].

²Results can be extended to calibrated D-branes in a non-supersymmetric setting [28].

³We thank D. Giataganas for discussion on this point.

metry versus non-Abelian T-duality in a general setting. Given the possibility of non-locally realized supersymmetries at the worldsheet level ⁴, we will resort to exploring the leading term in the low energy expansion. This issue was explored at this level in the Abelian set-up in [3], where it was shown for ten-dimensional $\mathcal{N} = 1$ supergravity that the Killing spinors had to be independent of the T-duality direction for supersymmetry to be unaffected by the duality. In this paper we will extend this Abelian T-duality result to type II ($\mathcal{N} = 2$) supergravity, and generalise it further to SU(2) non-Abelian T-duality in a (diagonal) Bianchi IX family of spacetimes. We will uncover a unifying picture. For both Abelian and non-Abelian T-duality, we will demonstrate that the supersymmetry conditions for the T-dual geometry can be written in terms of the supersymmetry conditions of the original geometry up to additional Kosmann spinorial Lie derivative [44] terms that vanish when supersymmetry is preserved. Therefore the Kosmann derivative encapsulates all information about supersymmetry breaking at leading order in the low energy expansion and provides an alternative way to determine preserved supersymmetry without an explicit knowledge of the Killing spinors of the original geometry.

The paper is organized as follows. Section 2 summarizes the Abelian case, emphasizing the role played by the Kosmann derivative. As the path is well-trodden, we restrict our analysis to massive IIA. In Section 3 we start our generalization to the SU(2) case ⁵. In this section we explore the Kosmann derivative in spacetimes with SU(2) isometry, to later, in section 4, introduce the NS sector of the SU(2) dual background and identify a preferred T-dual frame. In section 5 we work out the supersymmetry variations, and show that they are the same before and after duality provided the Killing spinors do not depend on the SU(2) directions in our preferred frame. We show that this condition is equivalent to the vanishing of the Kosmann derivative. We also generate the dual RR fluxes by imposing the matching of the gravitino supersymmetry variations and show that it is consistent with the remaining supersymmetry variations and the expected Bianchi identities. In section 6 we briefly discuss the extensions of the previous results to type IIB in non-Abelian cases. Section 7 contains some examples that illustrate the main results in the paper. These concern the SU(2) T-dual of the $\text{AdS}_5 \times Y^{p,q}$ geometries recently constructed in [19], the SU(2) T-dual of the $\text{AdS}_4 \times \mathbb{CP}^3$ background [18], and the new SU(2) T-dual of an $\text{AdS}_3 \times S^3 \times S^3 \times \mathbb{R}$ geometry [45] that we work out in this section. Section 8 contains our Conclusions, where we summarize the main results of the paper and discuss further prospects. Appendix A contains a brief account on squashed S^3 (Bianchi IX) spacetimes and Appendix B some technical details needed for the computation of the dual supersymmetry variations.

2 Abelian case

In this section we illustrate what happens to spacetime supersymmetry under Abelian T-duality. It is widely reported in the literature that if the Kosmann spinorial Lie derivative [44] vanishes, then supersymmetry will be unaffected by T-duality. A statement along these lines exists for type I supergravity in ten dimensions [3] and it is expected to generalise to

⁴Worldsheet supersymmetry for both Abelian and non-Abelian T-duality has been discussed in [2, 42, 43].

⁵In non-Abelian T-duality, the chirality of the T-dual theory depends on the dimension of the isometry group. SU(2) having three generators, flips chirality, just as in the U(1) case.

type II. Here we will show that if the Kosmann derivative vanishes, then supersymmetry is preserved under T-duality. As an immediate corollary, it follows that the Kosmann derivative determines the extent to which supersymmetry is broken when it is non-zero.

For concreteness, we confine our attention to a T-duality from massive IIA, but expect there to be no difference in the analysis in the context of type IIB. Much of the analysis here parallels that of [30], with the notable difference that we will be interested in the role of the Kosmann derivative, which will involve making it explicit in our supersymmetry analysis.

We consider a general spacetime with a $U(1)$ isometry:

$$\begin{aligned} ds_{10}^2 &= ds_9^2 + e^{2C}(dz + A_1)^2, \\ B_2 &= B + B_1 \wedge dz, \\ F_0 &= m, \\ F_2 &= G_2 + G_1 \wedge (dz + A_1), \\ F_4 &= G_4 + G_3 \wedge (dz + A_1), \end{aligned} \tag{2.1}$$

where m is the Romans' mass and we have allowed for various fields C, A_1, B_1 and G_p , $p = 1, 2, 3, 4$, which are defined on the transverse metric ds_9^2 . In addition, we consider a non-zero scalar dilaton, Φ and B is the transverse NS two-form, which will play no role in the transformation. A short calculation produces the spin connection for this spacetime:

$$\begin{aligned} \omega^\mu{}_\nu &= \bar{\omega}^\mu{}_\nu - \frac{1}{2}e^C F^\mu{}_\nu e^z, \\ \omega^z{}_\mu &= \partial_\mu C e^z + \frac{1}{2}e^C F_{\mu\rho} e^\rho, \end{aligned} \tag{2.2}$$

where $F = dA_1$ and $\mu = 0, \dots, 8$ denotes transverse directions. We are working in the natural frame where

$$e^\mu = \bar{e}^\mu, \quad e^z = e^C(dz + A_1). \tag{2.3}$$

The Lie derivative of a spinor with respect to a Killing direction K , or alternatively Kosmann derivative, may be defined as [44]

$$\mathcal{L}_K \eta = K^a \nabla_a \eta + \frac{1}{8}(dK)_{ab} \Gamma^{ab} \eta, \tag{2.4}$$

where $\nabla_a \equiv \partial_a + \frac{1}{4}\omega_{abc}\Gamma^{bc}$. Adopting $K = \partial_z$, it is an easy exercise to show that the spin connection and dK component cancel, leading to the relationship

$$\mathcal{L}_{\partial_z} \eta = \partial_z \eta. \tag{2.5}$$

As a result of this identity the Kosmann derivative with respect to ∂_z vanishes when the Killing spinors do not depend on the coordinate z

$$\mathcal{L}_{\partial_z} \eta = 0 \Leftrightarrow \partial_z \eta = 0. \tag{2.6}$$

To extract a projection condition from the Kosmann derivative, use may be made of the gravitino variation - see later (2.11) - in the z -direction to substitute for the covariant

derivative ⁶ :

$$\begin{aligned}\mathcal{L}_{\partial_z}\eta &= e^{2C}\frac{1}{8}\not{F}\eta + \frac{1}{2}e^C\not{\partial}C\Gamma^z\eta + \frac{1}{8}(\mathrm{d}B_1)_{\mu\nu}\Gamma^{\mu\nu}\sigma^3\eta - \frac{1}{8}e^{C+\Phi}\left[m\sigma^1\right. \\ &\quad \left.+ \frac{1}{2}\not{G}_2i\sigma^2 + e^{-C}\not{G}_1\Gamma^z i\sigma^2 + \frac{1}{24}\not{G}_4\sigma^1 + \frac{1}{6}e^{-C}\not{G}_3\Gamma^z\sigma^1\right]\Gamma_z\eta\end{aligned}\quad (2.7)$$

Here, we have used

$$\begin{aligned}H_3 &= \mathrm{d}B - \mathrm{d}B_1 \wedge A_1 + \mathrm{d}B_1 \wedge (\mathrm{d}z + A_1), \\ &= H + e^{-C}\mathrm{d}B_1 \wedge e^z.\end{aligned}\quad (2.8)$$

With the help of the T-duality conventions of Hassan [30], one can determine the T-dual geometry:

$$\begin{aligned}\mathrm{d}\hat{s}_{10}^2 &= \mathrm{d}s_9^2 + e^{-2C}(\mathrm{d}z + B_1)^2, \\ \hat{B}_2 &= B + A_1 \wedge (\mathrm{d}z + B_1) \\ e^{\hat{\Phi}} &= e^{\Phi-C}, \\ \hat{F}_1 &= m(\mathrm{d}z + B_1) - G_1, \\ \hat{F}_3 &= G_2 \wedge (\mathrm{d}z + B_1) - G_3, \\ \hat{F}_5 &= (1 + *_{10})G_4 \wedge (\mathrm{d}z + B_1).\end{aligned}\quad (2.9)$$

To ensure consistency, it is easy to confirm that the Bianchi identities give the same result before and after T-duality. This then is a check on the sign of the RR fluxes. We further notice that A_1 and B_1 are interchanged and the sign of C is flipped, making it easy to work out the T-dual spin connection from (2.2).

We are now in a position to identify how the supersymmetry variations transform. We start with some conventions. We recall the supersymmetry variations for type IIA [20, 30]

$$\delta\lambda = \frac{1}{2}\not{\partial}\Phi\eta - \frac{1}{24}\not{H}_3\sigma^3\eta + \frac{1}{8}e^\Phi\left[5m\sigma^1 + \frac{3}{2}\not{H}_2(i\sigma^2) + \frac{1}{24}\not{F}_4\sigma^1\right]\eta, \quad (2.10)$$

$$\delta\Psi_M = \nabla_M\eta - \frac{1}{8}H_{3MNP}\Gamma^{NP}\sigma^3\eta + \frac{1}{8}e^\Phi\left[F_0\sigma^1 + \frac{1}{2}\not{F}_2i\sigma^2 + \frac{1}{24}\not{F}_4\sigma^1\right]\Gamma_M\eta, \quad (2.11)$$

and type IIB

$$\delta\hat{\lambda} = \frac{1}{2}\not{\partial}\Phi\hat{\eta} - \frac{1}{24}\not{H}_3\sigma^3\hat{\eta} + \frac{1}{2}e^\Phi\left[\not{H}_1(i\sigma^2) + \frac{1}{12}\not{F}_3\sigma^1\right]\hat{\eta}, \quad (2.12)$$

$$\delta\hat{\Psi}_M = \nabla_M\hat{\eta} - \frac{1}{8}H_{3MNP}\Gamma^{NP}\sigma^3\hat{\eta} - \frac{1}{8}e^\Phi\left[\not{F}_1i\sigma^2 + \frac{1}{6}\not{F}_3\sigma^1 + \frac{1}{240}\not{F}_5i\sigma^2\right]\Gamma_M\hat{\eta}, \quad (2.13)$$

where η is a Majorana-Weyl spinor

$$\eta = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}. \quad (2.14)$$

⁶Where convenient we use slashed notation, where $\not{A} = A_{\mu_1\dots\mu_p}\Gamma^{\mu_1\dots\mu_p}$ for a p-form A .

We have introduced hats on the transformed Killing spinors to distinguish them from the Killing spinors before T-duality.

Substituting in our spacetime Ansatz (2.1), the transverse gravitino variation before T-duality may be expressed as

$$\begin{aligned}\delta\Psi_{\mu\pm} &= \nabla_\mu\epsilon_\pm + \frac{1}{4}e^CF_{\mu\nu}\Gamma^{\nu z}\epsilon_\pm \mp \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon_\pm \mp \frac{1}{4}(\mathrm{d}B_1)_{\mu\rho}e^{-C}\Gamma^{\rho z}\epsilon_\pm \\ &+ \frac{1}{8}e^\Phi\left[m \pm \not{G}_1e^{-C}\Gamma^z \pm \frac{1}{2}\not{G}_2 + \frac{1}{6}\not{G}_3e^{-C}\Gamma^z + \frac{1}{24}\not{G}_4\right]\Gamma_\mu\epsilon_\mp.\end{aligned}\quad (2.15)$$

After T-duality, one can again determine the T-dual gravitino variation, $\delta\hat{\Psi}_{\mu\pm}$, and one notes that they are not independent, provided we also redefine the Majorana-Weyl spinors,

$$\delta\hat{\Psi}_{\mu+} = -\Gamma^z\delta\Psi_{\mu+}, \quad \delta\hat{\Psi}_{\mu-} = \delta\Psi_{\mu-}, \quad \hat{\epsilon}_+ = -\Gamma^z\epsilon_+, \quad \hat{\epsilon}_- = \epsilon_-, \quad (2.16)$$

The choice of sign in the expressions above can be traced to an overall sign in the RR sector, which can be flipped by changing the sign of the fluxes. Note also that the chirality of ϵ_+ has been changed in the T-duality, as is expected.

There appears to be one small subtlety in this identification. Although the gravitino variations are structurally the same, the partial derivatives in the covariant derivative may not match. To see this, we write

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\bar{\omega}_{\mu\nu}\Gamma^{\mu\nu} - A_{1\mu}\partial_z, \quad \hat{\nabla}_\mu = \partial_\mu + \frac{1}{4}\bar{\omega}_{\mu\nu}\Gamma^{\mu\nu} - B_{1\mu}\partial_z, \quad (2.17)$$

which is not guaranteed to be the same when $\partial_z\eta \neq 0$. So, even at this early stage we notice that the variations only match when either the Killing spinors are independent of the T-duality direction, or if they are not, when both A_1 and B_1 are zero.

We next consider the direction on which we T-dualise. Before T-duality, the gravitino variation in this direction is zero. We have assumed this to rewrite the Kosmann derivative in (2.7). For this reason, the gravitino variation before T-duality may be expressed as

$$\delta\Psi_z = e^{-C}(\partial_z - \mathcal{L}_{\partial_z})\eta = 0, \quad (2.18)$$

where we have made use of (2.5). After T-duality, the variation may be expressed in terms of the Kosmann derivative as

$$\begin{aligned}\delta\hat{\Psi}_{z+} &= -e^C\Gamma^z\partial_z\epsilon_+ + \Gamma^ze^{-C}\mathcal{L}_{\partial_z}\epsilon_+, \\ \delta\hat{\Psi}_{z-} &= e^C\partial_z\epsilon_- + e^{-C}\mathcal{L}_{\partial_z}\epsilon_-, \end{aligned}\quad (2.19)$$

where we have used the earlier redefinitions of the spinors (2.16). Again, we see that supersymmetry will be preserved provided $\partial_z\eta = \mathcal{L}_{\partial_z}\eta = 0$.

Finally, we come to the dilatino variation (2.10). The dilatino variation before T-duality is

$$\begin{aligned}\delta\lambda_\pm &= \frac{1}{2}\not{\partial}\Phi\epsilon_\pm \mp \frac{1}{24}\not{H}\epsilon_\pm \mp \frac{1}{8}(\mathrm{d}B_1)_{\mu\nu}e^{-C}\Gamma^{\mu\nu}\Gamma^z\epsilon_\pm \\ &+ \frac{1}{8}\left[5m \pm \frac{3}{2}\not{G}_2 \pm 3\not{G}_1e^{-C}\Gamma^z + \frac{1}{6}\not{G}_3e^{-C}\Gamma^z + \frac{1}{24}\not{G}_4\right]\epsilon_\mp.\end{aligned}\quad (2.20)$$

Working out the analogous expressions for the T-dual dilatino variation, $\delta\hat{\lambda}$, we identify the following mappings:

$$\begin{aligned}\delta\hat{\lambda}_+ &= \Gamma^z\delta\lambda_+ + e^{-C}\mathcal{L}_{\partial_z}\epsilon_+, \\ \delta\hat{\lambda}_- &= \delta\lambda_- + e^{-C}\Gamma^z\mathcal{L}_{\partial_z}\epsilon_-, \end{aligned}\tag{2.21}$$

and arrive at the same conclusion as before.

In summary, starting with the most general Ansatz for a spacetime with U(1) isometry (2.1), we have shown that the supersymmetry variations before and after T-duality are the same provided the Killing spinors do not depend on the isometry direction. At some level this statement is intuitively obvious; if supersymmetry does not depend on a direction, it should be clear that we can perform a transformation on this direction and leave supersymmetry unaffected. However, by rewriting the supersymmetry variations, we have noted the special role of the Kosmann derivative. This means one can determine the amount of preserved supersymmetry by extracting a projection condition from the Kosmann derivative, or one can work out the Killing spinor of the original geometry explicitly and identify a subset of the supersymmetries that are independent of the T-duality directions.

3 Insights from Kosmann derivative

Before proceeding to the analogous analysis of the supersymmetry variations for SU(2) T-duality, it is prudent to first study the Kosmann derivative in this context. For spacetimes with SU(2) isometry there is an intrinsic choice in whether one uses left-invariant or right-invariant one-forms to parametrise the SU(2). We will opt to make use of left-invariant one-forms and will work with the metric

$$ds_{10}^2 = ds_7^2 + \sum_{a=1}^3 e^{2C_a}(\tau_a + A^a)^2,\tag{3.1}$$

where $d\tau_a = \frac{1}{2}\epsilon_{abc}\tau_b \wedge \tau_c$. Explicitly, the left-invariant one-forms may be expressed in terms of coordinates (θ, ϕ, ψ) in the accustomed fashion

$$\tau_1 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \quad \tau_2 = \cos\psi d\theta + \sin\psi \sin\theta d\phi, \quad \tau_3 = d\psi + \cos\theta d\phi.$$

Here A^a are SU(2)-valued gauge fields and C_a denote scalar warp factors and generically, when they are all different, the spacetime has a right-acting SU(2) isometry (see appendix A), when two are equal, the symmetry is enhanced to SU(2) \times U(1) and when all three are equal, we recover the SO(4) isometry of a round three-sphere. Gaugings of a round three-sphere in the context of non-Abelian T-duality were considered in [21]. Here we will focus on a more general case.

Employing a natural orthonormal frame,

$$e^\mu = \bar{e}^\mu, \quad e^a = e^{C_a}(\tau_a + A^a),\tag{3.2}$$

it is a straightforward exercise to work out the spin connection

$$\begin{aligned}
\omega_{\ 2}^1 &= \frac{1}{2}e^{-C_1-C_2-C_3}(e^{2C_1} + e^{2C_2} - e^{2C_3})e^3 - \frac{1}{2}e^{-C_1-C_2}(e^{2C_1} + e^{2C_2})A^3, \\
\omega_{\ \mu}^1 &= \partial_\mu C^1 e^1 - \frac{1}{2}e^{-C_1-C_2}(e^{2C_2} - e^{2C_1})A_\mu^3 e^2 + \frac{1}{2}e^{-C_1-C_3}(e^{2C_3} - e^{2C_1})A_\mu^2 e^3, \\
&+ \frac{1}{2}e^{C_1}F_{\mu\rho}^1 e^\rho, \\
\omega_{\ \nu}^\mu &= \bar{\omega}_{\ \nu}^\mu - \sum_a \frac{1}{2}e^{C_a}F_{\ \nu}^{a\mu}e^a,
\end{aligned} \tag{3.3}$$

where we have omitted cyclic terms in $a = 1, 2, 3$. As we have SU(2) gauge fields, it is not surprising to see non-Abelian field strengths, $F^a = dA^a + \frac{1}{2}\epsilon_{abc}A^b \wedge A^c$, appearing.

We will now show that the Kosmann derivative with respect to the right-invariant vectors K_a (A.2), dual to the right-invariant one-forms, vanishes when the spinor, η , is independent of these directions. The right-invariant vectors have the property that they commute with the left-invariant vector fields dual to the left-invariant one-forms parametrising our SU(2) isometry. Equivalently, we could write the metric in terms of right-invariant one-forms and consider the Kosmann derivative with respect to left-invariant vector fields.

For concreteness, we focus on K_3 , with similar statements holding for the remaining vectors. The first component in the spinorial derivative may be expressed as

$$K_3^a \nabla_a \eta = \nabla_\phi \eta = (e^{C_1} \cos \psi \sin \theta \nabla_1 + e^{C_2} \sin \psi \sin \theta \nabla_2 + e^{C_3} \cos \theta \nabla_3) \eta, \tag{3.4}$$

where ∇_ϕ denotes derivative with respect to the coordinate ϕ , while ∇_a , $a = 1, 2, 3$, denotes a covariant derivative with respect to left-invariant vectors (A.3).

Using the metric (3.1) and the corresponding natural orthonormal frame, we can determine the one-form dual to the vector K_3 ,

$$K_3 = \cos \psi \sin \theta e^{C_1} e^1 + \sin \psi \sin \theta e^{C_2} e^2 + \cos \theta e^{C_3} e^3. \tag{3.5}$$

Taking derivatives and making use of the following expressions,

$$\begin{aligned}
d\theta &= -\sin \psi (e^{-C_1} e^1 - A^1) + \cos \psi (e^{-C_2} e^2 - A^2), \\
d\phi &= \frac{\cos \psi}{\sin \theta} (e^{-C_1} e^1 - A^1) + \frac{\sin \psi}{\sin \theta} (e^{-C_2} e^2 - A^2), \\
d\psi &= -\cos \psi \frac{\cos \theta}{\sin \theta} (e^{-C_1} e^1 - A^1) - \sin \psi \frac{\cos \theta}{\sin \theta} (e^{-C_2} e^2 - A^2) \\
&+ (e^{-C_3} e^3 - A^3),
\end{aligned} \tag{3.6}$$

it is possible to show that the dK_3 breaks up into three parts

$$\begin{aligned}
\mathcal{L}_{K_3} \eta &= \cos \psi \sin \theta e^{C_1} P_1 \eta + \sin \psi \sin \theta e^{C_2} P_2 \eta + \cos \theta e^{C_3} P_3 \eta, \\
&= \partial_\phi \eta,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
P_1 \eta &= \nabla_1 \eta + \frac{1}{8}e^{C_1} \not{F}^1 \eta + \frac{1}{4}e^{-C_1-C_2-C_3}(e^{2C_1} - e^{2C_2} - e^{2C_3})\Gamma^{23} \eta \\
&+ \frac{1}{2}\not{C}_1 \Gamma^1 \eta - \frac{1}{4}e^{-C_1-C_3}(e^{2C_1} - e^{2C_3})\not{A}^2 \Gamma^3 \eta + \frac{1}{4}(e^{2C_1} - e^{2C_2})e^{-C_1-C_2}\not{A}^3 \Gamma^2 \eta,
\end{aligned} \tag{3.8}$$

and $P_2\eta, P_3\eta$ follow by cyclic symmetry in 1, 2, 3. Since the spin connection in $\nabla_a\eta$ cancels the remaining terms, $P_a\eta$ simply encode left-invariant vectors in our preferred frame acting on the spinor η .

One can repeat the above calculation to get similar expressions for $\mathcal{L}_{K_1}\eta$ and $\mathcal{L}_{K_2}\eta$ in terms of $P_a\eta$, before inverting to obtain the following:

$$\begin{aligned} e^{C_1}P_1\eta &= (-\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi)\mathcal{L}_{K_1}\eta - (\cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi)\mathcal{L}_{K_2}\eta \\ &\quad + \sin\theta\cos\psi\mathcal{L}_{K_3}\eta, \\ e^{C_2}P_2\eta &= (\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi)\mathcal{L}_{K_1}\eta + (\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi)\mathcal{L}_{K_2}\eta \\ &\quad + \sin\theta\sin\psi\mathcal{L}_{K_3}\eta, \\ e^{C_3}P_3\eta &= \sin\theta(-\cos\phi\mathcal{L}_{K_1} + \sin\phi\mathcal{L}_{K_2})\eta + \cos\theta\mathcal{L}_{K_3}\eta. \end{aligned} \quad (3.9)$$

It is now easy to convince oneself that

$$\mathcal{L}_{K_a}\eta = 0 \Leftrightarrow P_a\eta = 0 \Leftrightarrow \eta \text{ constant w. r. t. } (\theta, \phi, \psi). \quad (3.10)$$

As we will see later, the supersymmetry variations after T-duality can be rewritten to make the $P_a\eta$ manifest and they encode supersymmetry breaking during non-Abelian T-duality. More explicitly, focussing on the K_1 direction, we get:

$$\begin{aligned} P_1\eta &= \frac{1}{8}e^{C_1}\not{F}^1\eta + \frac{1}{4}e^{-C_1-C_2-C_3}(e^{2C_1} - e^{2C_2} - e^{2C_3})\Gamma^{23}\eta - \frac{1}{4}e^{-C_1-C_3}(e^{2C_1} - e^{2C_3})\not{A}^2\Gamma^3\eta \\ &\quad + \frac{1}{4}(e^{2C_1} - e^{2C_2})e^{-C_1-C_2}\not{A}^3\Gamma^2\eta + \frac{1}{2}\not{C}_1\Gamma^1\eta \\ &\quad + \frac{1}{8}H_{31NP}\Gamma^{NP}\sigma^3\eta - \frac{1}{8}e^\Phi\left[m\sigma^1 + \frac{1}{2}\not{F}_2(i\sigma^2) + \frac{1}{24}\not{F}_4\sigma^1\right]\Gamma_1\eta, \end{aligned} \quad (3.11)$$

where explicit expressions for H_3 , F_2 and F_4 will appear later. Already this is looking considerably more complicated than the Abelian case (2.7), but the expressions for P_2, P_3 follow from cyclic symmetry in 1, 2, 3.

4 Non-Abelian T-duality

In addition to determining the exact role of the Kosmann derivative in supersymmetry breaking under non-Abelian T-duality, a secondary goal of this work is to provide closed expressions for the SU(2) transformation directly in supergravity. This will provide a needed generalisation of the preliminary results of Ref. [20], which are only valid for spacetimes with SO(4) symmetry. In the future it will be possible to compare the seed solution to our SU(2) Ansatz, read off the data and produce the T-dual geometry.

In this section, we will focus on the NS sector and leave the analysis of the RR fluxes until the next section, where the transformed RR fluxes will be deduced from the supersymmetry conditions. The SU(2) T-duality rules we use can be found in [10], which we review now. Given a spacetime metric and NS two-form,

$$\begin{aligned} ds_{10}^2 &= G_{\mu\nu}dx^\mu dx^\nu + 2G_{\mu a}dx^\mu\tau_a + g_{ab}\tau_a\tau_b, \\ B &= \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu + B_{\mu a}dx^\mu \wedge \tau_a + \frac{1}{2}b_{ab}\tau_a \wedge \tau_b, \end{aligned} \quad (4.1)$$

where τ_a denote Maurer-Cartan one-forms, one combines the symmetric and anti-symmetric tensors in the usual way:

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad Q_{\mu a} = G_{\mu a} + B_{\mu a}, \quad Q_{a\mu} = G_{a\mu} + B_{a\mu}, \quad q_{ab} = g_{ab} + b_{ab}. \quad (4.2)$$

The T-dual symmetric and anti-symmetric tensors then follow from [10],

$$\begin{aligned} \hat{Q}_{\mu\nu} &= Q_{\mu\nu} - Q_{\mu a} M_{ab}^{-1} Q_{b\nu}, & \hat{q}_{ab} &= M_{ab}^{-1}, \\ \hat{Q}_{\mu a} &= Q_{\mu b} M_{ba}^{-1}, & \hat{Q}_{a\mu} &= -M_{ab}^{-1} Q_{b\mu}, \end{aligned} \quad (4.3)$$

where $M_{ab} = q_{ab} + f_{ab}{}^c v_c$ is the original tensor in the T-dual directions shifted with respect to the structure constants f_{abc} of the group, further contracted into the Lagrange multiplier v_c . We recall that v_c usually impose that the gauge field in the gauging procedure in [49] (see also [48]) is pure gauge, and eventually correspond to the T-dual coordinates. The dilaton receives a one-loop quantum correction analogous to the Abelian case:

$$\hat{\Phi} = \Phi - \frac{1}{2} \ln \det M \quad (4.4)$$

Specialising to our SU(2) case, where τ_a now correspond to left-invariant one-forms and $f_{abc} = \pm \epsilon_{abc}$ ⁷, we can identify the symmetric tensors $G_{\mu\nu}$, $G_{\mu a}$ and g_{ab} by comparing to our spacetime Ansatz (3.1):

$$G_{\mu\nu} = g_{\mu\nu} + \sum_{a=1}^3 e^{2C_a} A_\mu^a A_\nu^a, \quad G_{\mu a} = e^{2C_a} A_\mu^a, \quad g_{ab} = e^{2C_a} \delta_{ab}, \quad (4.5)$$

where $g_{\mu\nu}$ corresponds to the metric on the transverse seven-dimensional spacetime. In addition, we need to define an NS two-form, which we will take to be

$$B_2 = B + (B^a + db_a) \wedge \tau_a + \frac{1}{2} \epsilon_{abc} b_a \tau_b \wedge \tau_c, \quad (4.6)$$

where B , B^a and b_a are respectively, a two-form, three one-forms and three scalars and we have shifted B^a for convenience. As explained in [10], we can rewrite B_2 as

$$B_2 = B + B^a \wedge \tau_a + d(b_a \tau_a), \quad (4.7)$$

making it clear that the contribution from the scalars b_a is a pure gauge contribution⁸. Thus, b_a will not affect the EOMs and we could set them to zero from the offset. Still, this may preclude solutions, so we retain them in what follows. In the absence of b_a , the Lagrange multipliers v_a correspond to T-dual coordinates and get shifted to $z_a = v_a + b_a$ when the B-field is included. For later convenience, we record the field strength in our preferred frame:

$$H_3 = H + DB^a \wedge e^{-C_a} e^a - \frac{1}{2} \epsilon_{bc}^a e^{-C_b - C_c} B^a \wedge e^{bc}, \quad (4.8)$$

⁷The sign we use is irrelevant, since we can always change the sign of v_a .

⁸This Ansatz also precludes a field strength $H_3 \propto \tau_1 \wedge \tau_2 \wedge \tau_3$, a term that is ruled out when one attempts to gauge SU(2) isometries [47].

where we have defined

$$\begin{aligned} DB^a &= dB^a + \epsilon_{bc}^a A^b \wedge B^c, \\ H &= dB - dB^a \wedge A^a - \frac{1}{2} \epsilon_{abc} B^a \wedge A^b \wedge A^c. \end{aligned} \quad (4.9)$$

Throughout this work the covariant derivative of an $SU(2)$ -valued p-form, C_p^a will be uniformly defined to be $DC_p^a = dC_p^a + \epsilon_{bc}^a A^b \wedge C_p^c$.

It is then easy to read off the contribution to the tensors:

$$B_{\mu\nu} = B_{\mu\nu}, \quad B_{\mu a} = B_\mu^a + \partial_\mu b_a, \quad b_{ab} = \epsilon_{abc} b_c. \quad (4.10)$$

Once the tensors of the original spacetime Ansatz have been determined, the next step is to invert M_{ab}

$$M_{ab}^{-1} = \frac{1}{\Delta} \begin{pmatrix} e^{2(C_2+C_3)} + z_1^2 & z_1 z_2 - e^{2C_3} z_3 & z_1 z_3 + e^{2C_2} z_2 \\ z_1 z_2 + e^{2C_3} z_3 & e^{2(C_1+C_3)} + z_2^2 & z_2 z_3 - e^{2C_1} z_1 \\ z_1 z_3 - e^{2C_2} z_2 & z_2 z_3 + e^{2C_1} z_1 & e^{2(C_1+C_2)} + z_3^2 \end{pmatrix}, \quad (4.11)$$

where

$$\Delta = e^{2(C_1+C_2+C_3)} + e^{2C_1} z_1^2 + e^{2C_2} z_2^2 + e^{2C_3} z_3^2. \quad (4.12)$$

It is a straightforward task to now read off the T-dual metric from (4.3)

$$d\hat{s}^2 = \frac{1}{\Delta} \left[(z_1 Dz_1 + z_2 Dz_2 + z_3 Dz_3)^2 + e^{2(C_2+C_3)} Dz_1^2 + e^{2(C_1+C_3)} Dz_2^2 + e^{2(C_1+C_2)} Dz_3^2 \right],$$

where the transverse seven-dimensional spacetime comes along for the ride and the covariant derivatives encapsulate the $SU(2)$ gauge fields and the vector components of the original NS two-form

$$Dz_a = dz_a + B^a - \epsilon_{abc} z_b A^c. \quad (4.13)$$

The T-dual NS two-form

$$\begin{aligned} \hat{B}_2 &= B - \frac{1}{\Delta} (e^{2C_1} z_1 Dz_2 \wedge Dz_3 + e^{2C_2} z_2 Dz_3 \wedge Dz_1 + e^{2C_3} z_3 Dz_1 \wedge Dz_2) - Dz_1 \wedge A_1 \\ &\quad - Dz_2 \wedge A_2 - Dz_3 \wedge A_3 - z_1 A_2 \wedge A_3 - z_2 A_3 \wedge A_1 - z_3 A_1 \wedge A_2, \end{aligned} \quad (4.14)$$

incorporates the original transverse NS two-form, B , which is unaffected, in addition to generating new terms. The transformed dilaton may be written as

$$e^{\hat{\Phi}} = e^{\Phi} \Delta^{-\frac{1}{2}}. \quad (4.15)$$

Using the results of Ref. [10], one can further identify a preferred T-dual frame⁹ and check that it squares to recover the T-dual metric,

$$\hat{e}_-^a = e^{C_a} \Delta^{-1} \left[z_a z_c Dz_c + e^{2\sum_{b \neq a} C_b} Dz_a + \epsilon_{abc} z_b e^{2C_b} Dz_c \right]. \quad (4.16)$$

⁹This corresponds to \hat{e}_- in the notation of [10]. Switching frames to \hat{e}_+ simply involves flipping the sign of z_a in the generated solution. This action does not affect the metric, but changes the sign of the generated NS two-form.

We will henceforth drop subscripts on the frame.

At this stage, we could isolate a radial direction, r , by defining $z_a = r\mu_a$, where μ_a would denote constrained scalars on an S^2 . This approach was adopted in [21], where it led to a natural $SU(2)$ gauging of the S^2 . Here we will not consider the rewriting and will choose to retain the z_a as T-dual coordinates.

It is timely to record expressions for the corresponding derivatives

$$\begin{aligned}\hat{\partial}_a &= [e^{C_a}\partial_{z_a} + \epsilon_{abc}e^{-C_a}z_b\partial_{z_c}], \\ \hat{\partial}_\mu &= \partial_\mu - \epsilon_{abc}A_\mu^a z_b\partial_{z_c} - B_\mu^a\partial_{z_a},\end{aligned}\tag{4.17}$$

which allows us to rewrite Dz_a in terms of the frame,

$$Dz_a = e^{C_a}\hat{e}^a - \epsilon_{abc}z_b e^{-C_c}\hat{e}^c.\tag{4.18}$$

5 Supersymmetry: IIA to IIB

In this section, we will address supersymmetry in the context of an $SU(2)$ non-Abelian T-duality by studying the transformation of the supersymmetry variations. As we shall see in due course, the variations before and after T-duality are the same provided the Killing spinors are independent of the coordinates of the $SU(2)$ factor on which we dualise. This condition is equivalent to the vanishing of the Kosmann derivative, an observation that reduces the problem of preserved supersymmetry to either an explicit determination of the Killing spinors or an evaluation of the Kosmann derivative.

By analogy with the Abelian case, for example [30], or the analysis of section 2, it is expected that the Majorana-Weyl Killing spinors transform as follows

$$\tilde{\epsilon}_- = \epsilon_-, \quad \tilde{\epsilon}_+ = \beta\Omega^{-1}\epsilon_+, \tag{5.1}$$

where $\beta = \pm 1$ allows for an ambiguity in signs, which will be fixed by combing the NS sector with the RR sector, and Ω takes the form [10]

$$\Omega = \frac{1}{\sqrt{\Delta}} (-\alpha e^{C_1+C_2+C_3}\Gamma^{123} + z_a e^{C_a}\Gamma^a). \tag{5.2}$$

Again $\alpha = \pm 1$ allows for a choice of sign, which will get fixed by the T-dual NS two-form (4.14). Since Ω has an odd number of gamma matrices, it is clear that it flips the chirality of the spinors. We will adopt $\Gamma^{0123456789}\tilde{\eta} = -\tilde{\eta}$ in type IIB.

Now our strategy is morally the same as [30]. We will use our knowledge of the transformed NS sector as a means to generate the transformation of the RR fluxes. We will do this by initially comparing the gravitino variation along the transverse seven-dimensional spacetime, $\delta\Psi_\mu$. We will then check that the resulting fluxes, when plugged back into the remaining variations, either produce Kosmann derivative terms (3.9), (3.11) or the supersymmetry variations of the original spacetime Ansatz. As a further consistency check, we will confirm that the Bianchi identities (appendix C) before and after T-duality agree. One could go further and show that the EOMs match, but it is a given that supersymmetry

usually implies the EOMs through integrability and we leave a more elegant proof of this to future work ¹⁰.

We have used the SU(2) T-duality rules (4.3) to generate the NS sector. Before T-duality, the gravitino variation may be expressed as

$$\begin{aligned}\delta\Psi_{\mu\pm} &= \nabla_\mu\epsilon_\pm - \frac{1}{4}\epsilon_{abc}A_\mu^a e^{C_b-C_c}\Gamma^{bc}\epsilon_\pm + \frac{1}{4}e^{C_a}F_{\mu\nu}^a\Gamma^{\nu a}\epsilon_\pm \mp \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon_\pm \mp \frac{1}{4}D_\mu B_\rho^a e^{-C_a}\Gamma^{\mu a}\epsilon_\pm \\ &\pm \frac{1}{8}\epsilon_{abc}B^a e^{-C_b-C_c}\Gamma^{bc}\epsilon_\pm + \frac{1}{8}e^\Phi\left[F_0 \pm \frac{1}{2}\not{F}_2 + \frac{1}{24}\not{F}_4\right]\Gamma_\mu\epsilon_{\mp}.\end{aligned}\quad (5.3)$$

In deriving this expression, we have made use of the spin connection (3.3) and the field strength H_3 (4.8). So far we have not specified an Ansatz for the RR sector, so $F_p, p = 0, 2, 4$ appear.

Repeating the analysis with the T-dual NS sector, given by expressions (B.1) and a spin connection that can be derived from (B.1), after various cancellations, the T-dual gravitino variation may be massaged into the form

$$\begin{aligned}\delta\hat{\Psi}_{\mu-} &= \nabla_\mu\epsilon_- - \frac{1}{4}\epsilon_{abc}A_\mu^a e^{C_b-C_c}\Gamma^{bc}\epsilon_- + \frac{1}{4}e^{C_a}F_{\mu\nu}^a\Gamma^{\nu a}\epsilon_- + \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon_- \\ &+ \frac{1}{4}D_\mu B_\rho^a e^{-C_a}\Gamma^{\mu a}\epsilon_- - \frac{1}{8}\epsilon_{abc}B^a e^{-C_b-C_c}\Gamma^{bc}\epsilon_-, \end{aligned}\quad (5.4)$$

and

$$\begin{aligned}\beta\Omega\delta\hat{\Psi}_{\mu+} &= \left(\nabla_\mu\epsilon_+ - \frac{1}{4}\epsilon_{abc}A_\mu^a e^{C_b-C_c}\Gamma^{bc}\epsilon_+ + \frac{1}{4}e^{C_a}F_{\mu\nu}^a\Gamma^{\nu a}\epsilon_+ - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon_+ \right. \\ &\left. - \frac{1}{4}D_\mu B_\rho^a e^{-C_a}\Gamma^{\mu a}\epsilon_+ + \frac{1}{8}\epsilon_{abc}B^a e^{-C_b-C_c}\Gamma^{bc}\epsilon_+\right), \end{aligned}\quad (5.5)$$

where we have re-expressed the T-dual spinor, $\tilde{\eta}$ in terms of the original spinor η . We observe that the transverse gravitino variation, when restricted to the NS sector, agrees precisely before and after T-duality, providing reassuring confirmation that the SU(2) T-duality rules have been applied correctly. The choice of α in (5.2) is determined by the sign of the NS two-form (4.14) and gets fixed to $\alpha = +1$.

Now that we have some confidence that the NS sector has been T-dualised correctly, we can, given an Ansatz for the original RR sector, generate the T-dual RR sector simply by coercing the supersymmetry variations to match. We are thus free to consider a general Ansatz for the RR sector, one with SU(2) symmetry:

$$\begin{aligned}F_0 &= m, \\ F_2 &= G_2 + J_1^a \wedge (\tau_a + A^a) + \frac{1}{2}\epsilon_{abc}K_0^a(\tau_b + A^b) \wedge (\tau_c + A^c), \\ F_4 &= G_4 + L_3^a \wedge (\tau_a + A^a) + \frac{1}{2}\epsilon_{abc}M_2^a \wedge (\tau_b + A^b) \wedge (\tau_c + A^c) \\ &\quad + N_1 \wedge (\tau_1 + A^1) \wedge (\tau_2 + A^2) \wedge (\tau_3 + A^3),\end{aligned}\quad (5.6)$$

¹⁰A partial direct proof using consistent dimensional reduction can be found in [21], but all C_a are equal, i. e. there is a round S^3 , and the flux Ansatz is not completely general.

where repeated indices are summed and we have expanded in left-invariant one-forms, which appear wedged with forms living on the transverse seven-dimensional spacetime, G_p, J_1, K_0, L_3, M_2 and N_1 . m is a constant corresponding to the Romans' mass. In the context of non-Abelian T-duality, an RR flux Ansatz comprising the F_4 term expanded in (right-invariant) one-forms appeared previously in Ref. [21], but a thorough supersymmetry analysis was not presented. Here we generalise the Ansatz of Ref. [21] to include an expansion for F_2 . Setting all fields except m, G_2 and G_4 to zero, one recovers the $\text{SO}(4)$ -invariant Ansatz of Ref. [20].

We can now generate the correct T-dual for the RR sector:

$$\begin{aligned}
\hat{F}_1 &= m z_a e^{C_a} \hat{e}^a - z_a J_1^a - K_0^a e^{C_a} \hat{e}^a + \epsilon_{abc} K_0^a z_b e^{-C_c} \hat{e}^c + N_1, \\
\hat{F}_3 &= m e^{\sum_a C_a} \hat{e}^{123} + e^{\sum C_a} *_7 G_4 + G_2 \wedge z_a e^{C_a} \hat{e}^a - \frac{1}{2} \epsilon_{abc} J_1^a \wedge e^{C_b+C_c} \hat{e}^{bc} \\
&\quad + J_1^a \wedge e^{-C_a} \hat{e}^a \wedge z_b e^{C_b} \hat{e}^b + z_a K_0^a e^{2C_a} e^{-\sum_b C_b} \hat{e}^{123} - N_1 \wedge \frac{1}{2} \epsilon_{abc} z_a e^{-C_b-C_c} \hat{e}^{bc} \\
&\quad - z_a L_3^a - M_2^a \wedge e^{C_a} \hat{e}^a + \epsilon_{abc} M_2^a z_b \wedge e^{-C_c} \hat{e}^c, \\
\hat{F}_5 &= (1 + *_{10}) \left[G_4 \wedge z_a e^{C_a} \hat{e}^a + e^{\sum C_a} G_2 \wedge \hat{e}^{123} - \frac{1}{2} \epsilon_{abc} L_3^a \wedge e^{C_b+C_c} \hat{e}^{bc} \right. \\
&\quad \left. + L_3^a \wedge e^{-C_a} \hat{e}^a \wedge z_b e^{C_b} \hat{e}^b + z_a M_2^a e^{2C_a} \wedge e^{-\sum_b C_b} \hat{e}^{123} \right], \tag{5.7}
\end{aligned}$$

where again repeated indices are summed and we have set $\beta = -1$ for consistency. One could alternatively generate the RR sector as initially suggested in [7] by acting on the flux bi-spinor with Ω^{-1} , however one is still left the task of confirming the EOMs and the supersymmetry variations and matching the supersymmetry variations directly constitutes an advantageous short-cut.

We now check that our choice of RR sector is consistent with the remaining supersymmetry variations, starting with the gravitino variations along the T-dual directions. Before proceeding, we will explicitly write out an expression for P_1 . To do this, we recall (3.11) and insert in the explicit form of H_3 from (4.8) and F_2, F_4 from (5.6). This leads to the following lengthy expression:

$$\begin{aligned}
P_1 \eta &= \frac{1}{8} e^{C_1} \not{F}^1 \eta + \frac{1}{4} e^{-C_1-C_2-C_3} (e^{2C_1} - e^{2C_2} - e^{2C_3}) \Gamma^{23} \eta - \frac{1}{4} e^{-C_1-C_3} (e^{2C_1} - e^{2C_3}) \not{A}^2 \Gamma^3 \eta \\
&\quad + \frac{1}{4} (e^{2C_1} - e^{2C_2}) e^{-C_1-C_2} \not{A}^3 \Gamma^2 \eta + \frac{1}{2} \not{C} C_1 \Gamma^1 \eta + \frac{1}{8} \not{D} B^1 e^{-C_1} \sigma^3 \eta \\
&\quad + \frac{1}{4} e^{-C_1-C_2} \not{B}^3 \Gamma^2 \sigma^3 \eta - \frac{1}{4} e^{-C_1-C_3} \not{B}^2 \Gamma^3 \sigma^3 \eta - \frac{1}{8} e^\Phi \left[m \sigma^1 + \frac{1}{2} \not{C}^2 (i \sigma^2) \right. \\
&\quad + \not{J}_1^a e^{-C_a} \Gamma^a (i \sigma^2) + \frac{1}{2} \epsilon_{abc} K_0^a e^{-C_b-C_c} \Gamma^{bc} (i \sigma^2) + \frac{1}{24} \not{G}_4 \sigma^1 + \frac{1}{6} \not{L}_3^a e^{-C_a} \Gamma^a \sigma^1 \\
&\quad \left. + \frac{1}{4} \epsilon_{abc} M_2^a e^{-C_b-C_c} \Gamma^{bc} \sigma^1 + N_1 e^{-\sum_a C_a} \Gamma^{123} \right] \Gamma_1 \eta, \tag{5.8}
\end{aligned}$$

where repeated indices are assumed to be summed.

After T-duality, we find that the gravitino variations in the T-dual directions can be

written in terms of $P_a\eta$:

$$\begin{aligned}\delta\hat{\Psi}_{1-} &= \hat{\partial}_1\epsilon_- - P_1\epsilon_- + \frac{2}{\Delta}e^{C_1}\left[(e^{2C_2+2C_3} + z_1^2)e^{C_1}P_1\epsilon_- \right. \\ &\quad \left. - (e^{2C_3}z_3 - z_1z_2)e^{C_2}P_2\epsilon_- + (e^{2C_2}z_2 + z_1z_3)e^{C_3}P_3\epsilon_- \right], \\ \beta\Omega\delta\hat{\Psi}_{1+} &= \hat{\partial}_1\epsilon_+ - P_1\epsilon_+.\end{aligned}\tag{5.9}$$

As expected we see that the gravitino variation along a representative T-dual direction vanishes provided the Kosmann derivative is zero, $\mathcal{L}_{K_a}\eta = 0$, and the derivative with respect to the T-dual directions (4.17), ∂_{z_a} , is also zero. Since our Killing spinors did not initially depend on these directions and we have introduced them in the process of performing the T-duality, this latter condition is trivially satisfied. Observe also that due to the cyclic symmetry, it is enough to check one of the internal directions.

We finally come to the dilatino variation. A similar calculation to above reveals that

$$\begin{aligned}\delta\hat{\lambda}_- &= \delta\lambda_- \\ &+ \frac{1}{\Delta}\left[(e^{2C_2+2C_3} + z_1^2)e^{C_1}\Gamma^1 + (e^{2C_3}z_3 + z_1z_2)e^{C_2}\Gamma^2 - (e^{2C_2}z_2 - z_1z_3)e^{C_3}\Gamma^3\right]e^{C_1}P_1\epsilon_- \\ &+ \frac{1}{\Delta}\left[(e^{2C_3+2C_1} + z_2^2)e^{C_2}\Gamma^2 + (e^{2C_1}z_1 + z_2z_3)e^{C_3}\Gamma^3 - (e^{2C_3}z_3 - z_1z_2)e^{C_1}\Gamma^1\right]e^{C_2}P_2\epsilon_- \\ &+ \frac{1}{\Delta}\left[(e^{2C_1+2C_2} + z_3^2)e^{C_3}\Gamma^3 + (e^{2C_2}z_2 + z_1z_3)e^{C_1}\Gamma^1 - (e^{2C_1}z_1 - z_2z_3)e^{C_2}\Gamma^2\right]e^{C_3}P_3\epsilon_- \\ &= \delta\lambda_- + \sum_{a=1}^3 \frac{1}{2}\Gamma^a \left[\delta\hat{\Psi}_{a-} + P_a\epsilon_-\right],\end{aligned}\tag{5.10}$$

where we have made use of (5.9), various cyclic expressions and $\hat{\partial}_a\eta = 0$ to rewrite the above expression. Similarly, one can express $\delta\hat{\lambda}_+$ in terms of $P_a\eta$:

$$\begin{aligned}\beta\Omega\delta\hat{\lambda}_+ &= \delta\lambda_+ \\ &- \frac{1}{\Delta}\left[(e^{2C_2+2C_3} + z_1^2)e^{C_1}\Gamma^1 - (e^{2C_3}z_3 - z_1z_2)e^{C_2}\Gamma^2 + (e^{2C_2}z_2 + z_1z_3)e^{C_3}\Gamma^3\right]e^{C_1}P_1\epsilon_+ \\ &- \frac{1}{\Delta}\left[(e^{2C_3+2C_1} + z_2^2)e^{C_2}\Gamma^2 - (e^{2C_1}z_1 - z_2z_3)e^{C_3}\Gamma^3 + (e^{2C_3}z_3 + z_1z_2)e^{C_1}\Gamma^1\right]e^{C_2}P_2\epsilon_+ \\ &- \frac{1}{\Delta}\left[(e^{2C_1+2C_2} + z_3^2)e^{C_3}\Gamma^3 - (e^{2C_2}z_2 - z_1z_3)e^{C_1}\Gamma^1 + (e^{2C_1}z_1 + z_2z_3)e^{C_2}\Gamma^2\right]e^{C_3}P_3\epsilon_+.\end{aligned}\tag{5.11}$$

Note that up to a change in sign in terms even under $z_a \leftrightarrow -z_a$, the above expression is the same as (5.10).

This brings our rewriting of the supersymmetry variations to an end. In all equations we have noted that the Killing spinor equations after an $SU(2)$ non-Abelian T-duality can be written in terms of the Killing spinor equations before T-duality up to the vanishing of $P_a\eta$.

As discussed in the previous section, $P_a\eta = 0$ is equivalent to the vanishing of the Kosmann derivative with respect to the (right) $SU(2)$ isometry group. This condition further implies that the Killing spinors are constant with respect to these directions.

Therefore the problem of establishing the extent of supersymmetry breaking under non-Abelian T-duality boils down to an analysis of the Killing spinors themselves. One can either break supersymmetry by imposing suitable projection conditions, thus imposing that the Killing spinors are constant with respect to the T-duality directions, or one can analyse the Kosmann derivative to extract projection conditions. This observation explains why supersymmetry is broken in the background $AdS_5 \times S^5$ under an $SO(6)$ transformation [46]; there are no Killing spinors that are constant with respect to the action of the full $SO(6)$ isometry.

6 IIB to IIA

In this section we repeat the process of the section 5, except this time describing the transformation from type IIB to type IIA supergravity. More concretely, we will adopt the same results for the NS sector, impose the same Ansatz for the Killing spinor transformation (5.1) and use supersymmetry to extract the transformed RR fluxes given an initial flux Ansatz. We will assume that there is no deviation from the analysis of the previous section and that one can indeed show that transformed Killing spinor equations correspond to the original Killing spinor equations modulo the appearance of the Kosmann derivative.

We consider a general RR flux Ansatz with a right-acting $SU(2)$ symmetry. In particular, we write this as ¹¹

$$\begin{aligned} F_1 &= G_1, \\ F_3 &= G_3 + X_2^a \wedge (\tau_a + A^a) + \frac{1}{2} \epsilon_{abc} Y_1^a \wedge (\tau_b + A^b) \wedge (\tau_c + A^c) \\ &\quad + m(\tau_1 + A^1) \wedge (\tau_2 + A^2) \wedge (\tau_3 + A^3), \\ F_5 &= (1 + *_{10}) \left[Z_4^a \wedge (\tau_a + A^a) + G_2 \wedge (\tau_1 + A^1) \wedge (\tau_2 + A^2) \wedge (\tau_3 + A^3) \right], \end{aligned} \quad (6.1)$$

where m is a suggestive constant and we have used G_p , $p = 1, 2, 3$, to label the $SO(4)$ -singlets that appeared in Ref. [20]. The remaining fields, X_2, Y_1 and Z_4 are novel to the analysis presented in this paper. Owing to self-duality of the five-form flux, terms involving an even number of products of the base one-form $(\tau_a + A^a)$ are implied. There are numerous explicit examples of $SU(2)$ transformations covered by our Ansatz in the literature [7, 10, 19, 23, 24].

¹¹The inclusion of an $SU(2)$ one-form in F_1 is inconsistent with the Bianchi $dF_1 = 0$.

From an analysis of the gravitino variation, $\delta\Psi_\mu$, one can read off the T-dual RR flux:

$$\begin{aligned}
\hat{F}_0 &= m, \\
\hat{F}_2 &= z_a e^{C_a} \hat{e}^a \wedge G_1 + z_a X_2^a - Y_1^a \wedge e^{C_a} \hat{e}^a - \epsilon_{abc} Y_1^a \wedge z_b e^{-C_c} \hat{e}^c + m \frac{1}{2} \epsilon_{abc} z_a e^{-C_b - C_c} \hat{e}^{bc}, \\
&+ G_2, \\
\hat{F}_4 &= e^{\sum_b C_b} G_1 \wedge \hat{e}^{123} + z_a e^{C_a} \hat{e}^a \wedge G_3 + e^{\sum_b C_b} *_7 G_3 - \frac{1}{2} \epsilon_{abc} X_2^a \wedge e^{C_b + C_c} \hat{e}^{bc} \\
&- X_2^a e^{-C_a} \hat{e}^a \wedge z_b e^{C_b} \hat{e}^b - z_a e^{2C_a} Y_1^a \wedge e^{-\sum_b C_b} \hat{e}^{123} + z_a Z_4^a - e^{\sum_b C_b} *_7 Z_4^a \wedge e^{-C_a} \hat{e}_a \\
&- \epsilon_{abc} e^{-C_a} *_7 Z_4^a \wedge z_b e^{C_b} \hat{e}^c + G_2 \wedge \frac{1}{2} \epsilon_{abc} z_a e^{-C_b - C_c} \hat{e}^{bc}.
\end{aligned} \tag{6.2}$$

As expected, m clearly corresponds to the Romans' mass. Combined with earlier results in this paper, this gives a complete mapping for type IIB supergravity solutions with $SU(2)$ isometry.

7 Examples

The results of the previous sections provide a comprehensive proof that the question of supersymmetry preservation under $U(1)$ and $SU(2)$ T-duality boils down to checking the Kosmann derivative along the given isometry in the original solution. This is a powerful result with great utility and in this section we will illustrate this point with some recent and new examples.

We begin by showing how supersymmetry is preserved in recent solutions generated by non-Abelian T-duality acting on $AdS_5 \times Y^{p,q}$ and $AdS_4 \times \mathbb{CP}^3$ [18, 19]. We then generate a new AdS_3 solution in type IIB preserving 16 supercharges and leave a discussion of further examples for concluding remarks. In the explicit examples we study we do not present the form of the dual Killing spinor, as in each case, this is defined in terms of the spinor of the original geometry (5.1).

Let us start with a recap of the main result; there are essentially two cases:

1. Supersymmetry is preserved when the Kosmann derivative of the original Killing spinor η w. r. t the T-duality directions vanishes.
2. Supersymmetry is broken whenever the Kosmann derivative w. r. t. the T-duality directions is non-zero, with the degree of supersymmetry breaking determined by the Kosmann derivative.

As we have elucidated earlier, when the Kosmann derivative vanishes, the Killing spinors are independent of the isometry directions making it intuitively obvious that supersymmetry will be preserved. Therefore, where the original Killing spinors are known, or the initial geometry is suitably simple, it may be easier to directly examine the Killing spinors. Whenever this proves problematic, one still has recourse to the Kosmann derivative. The latter is of course frame independent, however it is useful to work in the preferred frame of T-duality of (2.3) or (3.2), since after all, one needs to be in this frame to read off the T-dual solution. This reduces to applying equations (4.14), (4.16) and either (5.7) or (6.2).

To illustrate our methods, let us first look at a simple case that preserves all the supercharges of the original solution, namely the $SU(2)$ transformation of $AdS_5 \times Y^{p,q}$. This example was recently presented in [19].

The Dual of $AdS_5 \times Y^{p,q}$

We recall that $Y^{p,q}$ is an infinite family [50] of Sasaki-Einstein manifolds that, when embedded in type-IIB supergravity, is dual to $\mathcal{N} = 1$ superconformal quivers in four dimensions [53, 54]. They can be realised as the near-horizon limit of a stack of N D3 branes placed at the tip of the associated Calabi-Yau cone.

In [19] this geometry was T-dualised along its $SU(2)$ isometry and it was shown that $\mathcal{N} = 1$ supersymmetry was preserved by demonstrating that an $SU(2)$ -structure exists on the internal space¹². It was additionally shown that the Kosmann derivative vanishes, implying directly through our earlier work that supersymmetry is preserved. For completeness, here we use results in the literature, e.g. [56], to show that the Killing spinor in the frame of equation (3.2) is independent of the isometry directions.

The Killing spinors of $AdS_5 \times Y^{p,q}$ were originally derived in [56] in the orthonormal frame ($c = 1$)

$$\begin{aligned} e^{x^\mu} &= r dx^\mu, & e^r &= \frac{dr}{r}, & e^y &= -\frac{dy}{\sqrt{wv}}, & e^\beta &= -\frac{\sqrt{wv}}{6}(d\beta - \cos\theta d\phi), \\ e^1 &= \frac{\sqrt{1-y}}{\sqrt{6}}d\theta, & e^2 &= \frac{\sqrt{1-y}}{\sqrt{6}}\sin\theta d\phi, & e^3 &= \frac{1}{3}(d\psi + yd\beta + (1-y)\cos\theta d\phi), \end{aligned} \quad (7.3)$$

where we have defined

$$w = \frac{2(a-y^2)}{1-y}, \quad v = \frac{a-3y^2+2y^3}{a-y^2}, \quad (7.4)$$

with a a constant. The Killing spinor is determined to be

$$\eta = e^{-\frac{i}{2}\psi} r^{-\frac{\Gamma_\star}{2}} \left(1 + \frac{\Gamma_r}{2} x^\mu \Gamma_\mu (1 + \Gamma_\star) \right) \eta_0 \quad (7.5)$$

where $\Gamma_\star = i\Gamma_{x^0 x^1 x^2 x^3}$ and η_0 is a constant spinor satisfying the projection conditions¹³

$$\Gamma_{y\beta}\eta_0 = -i\eta_0, \quad \Gamma_{12}\eta_0 = i\eta_0. \quad (7.6)$$

We observe that in this frame, the natural frame for Abelian T-duality with respect to the isometry ∂_ψ , the Killing spinors necessarily depend on ψ , so supersymmetry will be completely broken under Abelian T-duality. This is not the case for an $SU(2)$ T-duality, which we will show by exploiting a frame rotation to recast (7.3) in terms of a natural frame

¹²See [55] for the original work on G-structures in the presence of non trivial RR fields.

¹³Here η is a complex Killing spinor in type IIB. The real Majorana-Weyl spinors can be extracted via $\eta = \epsilon_1 + i\epsilon_2$. This change must be supplemented in the projections by the replacement $i\eta \rightarrow (i\sigma_2)\eta$.

for SU(2) T-duality [19]. More precisely, we consider the following rotations ¹⁴:

$$\begin{aligned}
e^{\alpha'} &= \cos \lambda e^\beta + \sin \lambda e^3 = \frac{1}{3} \frac{\sqrt{wv}}{\sqrt{g}} d\alpha, \\
e^{3'} &= \cos \lambda e^3 - \sin \lambda e^\beta = (\sqrt{g} \tau_3 + \frac{wf}{\sqrt{g}} d\alpha), \\
e^{1'} &= \cos \psi e^1 + \sin \psi e^2 = \frac{\sqrt{1-y}}{\sqrt{6}} \tau_1, \quad e^{2'} = \cos \psi e^2 - \sin \psi e^1 = \frac{\sqrt{1-y}}{\sqrt{6}} \tau_2, \\
e^{a'} &= e^a, \quad a = x^\mu, r, y,
\end{aligned} \tag{7.7}$$

where

$$f = \frac{a - 2y + y^2}{6(a - y^2)}, \quad \cot \lambda = \frac{2(1-y)}{\sqrt{wv}}, \quad g = \frac{v}{9} + wf^2. \tag{7.8}$$

Such rotations act on the spinor by a matrix \mathcal{S} which satisfies $\mathcal{S}^{-1} \Gamma_a \mathcal{S} = \Lambda_a^b \Gamma_b$. The solution for a rotation by χ acting on flat directions z_1, z_2 is simply

$$\mathcal{S}_\chi = e^{\frac{1}{2} \chi \Gamma_{z_1 z_2}}, \tag{7.9}$$

which results in a spinor independent of the SU(2) directions θ, ϕ, ψ , namely

$$\eta' = e^{\frac{1}{2} \lambda \Gamma'_{\alpha 3} r^{-\frac{\Gamma'_*}{2}}} \left(1 + \frac{\Gamma'_r}{2} x^\mu \Gamma'_\mu (1 + \Gamma'_*) \right) \eta_0 \tag{7.10}$$

Since the Killing spinor is independent of the SU(2) directions, it is now obvious that $\mathcal{N} = 1$ supersymmetry is preserved under the SU(2) transformation.

Adopting our notation and (7,3)-split for the spacetime, the solution may be expressed in the form (3.1) as

$$\begin{aligned}
ds_7^2 &= ds^2(AdS_5) + \frac{dy^2}{wv} + \frac{wv}{9g} d\alpha^2, \\
e^{2C_1} &= e^{2C_2} = \frac{1-y}{6}, \quad e^{2C_3} = g, \\
A^1 &= A^2 = 0, \quad A^3 = \frac{wf}{g} d\alpha.
\end{aligned} \tag{7.11}$$

Further comparison with (6.1) means the metric is supported by non trivial RR 5-form flux with

$$G_2 = -\frac{2}{9} (1-y) dy \wedge d\alpha. \tag{7.12}$$

All other terms in (6.1) are zero and the dilaton is a constant. One is now in a position to generate the solution using earlier results and we omit further details.

¹⁴In the process of performing these rotations we also redefine $\beta = -(6\alpha + \psi)$ and relabel the frame accordingly.

The Dual of $\text{AdS}_4 \times \mathbb{CP}^3$

$\text{AdS}_4 \times \mathbb{CP}^3$ is a solution in type IIA first introduced in [51] and later identified as the holographic description of $\mathcal{N} = 6$ Chern-Simons matter theory [52]. The geometry preserves 24 supersymmetries. To see this, one can decompose the general Killing spinor η as a product of four and six-dimensional spinors $\eta = \eta_{\text{AdS}_4} \otimes \eta_{\mathbb{CP}^3}$, where the standard spinor on AdS_4 , η_{AdS_4} preserves four supercharges. In contrast to the external spinor's simplicity, the internal spinor is generally quite complicated and depends on all coordinates on \mathbb{CP}^3 . Despite the complexity, it can be shown that \mathbb{CP}^3 preserves a 6-component spinor [51], giving a total of 24 preserved supersymmetries.

It has recently been shown [18] (see appendix B for details) that when the internal spinor has only two components, a frame exists

$$\begin{aligned} e^\zeta &= L d\zeta, & e^{\theta_1} &= \frac{L}{2} \cos \zeta d\theta_1, & e^{\phi_1} &= \frac{L}{2} \cos \zeta \sin \theta_1 d\phi_1, \\ e^1 &= \frac{L}{2} \sin \zeta \tau_1, & e^2 &= \frac{L}{2} \sin \zeta \tau_2, & e^3 &= \frac{L}{2} \sin \zeta \cos \zeta (\tau_3 + \cos \theta_1 d\phi_1), \end{aligned} \quad (7.13)$$

where the truncated Killing spinor depends only on two arbitrary constants a, b and may be expressed as¹⁵

$$\hat{\eta}_{\mathbb{CP}^3} = (0, a \cos \zeta, a \sin \zeta, 0, 0, -b \sin \zeta, b \cos \zeta, 0)^T \quad (7.15)$$

With this choice all dependence on the isometry directions drops out. Thus we find that performing an $\text{SU}(2)$ T-duality on $\text{AdS}_4 \times \mathbb{CP}^3$ preserves a total of 8 supercharges, resulting in $\mathcal{N} = 2$ supersymmetry in three dimensions.

The spacetime metric can be rewritten in terms of our Ansatz (3.1) as

$$\begin{aligned} ds_7^2 &= \frac{L^2}{4} ds^2(\text{AdS}_4) + L^2 \left[d\zeta^2 + \frac{1}{4} \cos^2 \zeta (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) \right], \\ e^{2C_1} &= e^{2C_2} = \frac{L^2}{4} \sin^2 \zeta, & e^{2C_3} &= \frac{L^2}{4} \sin^2 \zeta \cos^2 \zeta, \\ A^1 &= A^2 = 0, & A^3 &= \cos \theta_1 d\phi_1. \end{aligned} \quad (7.16)$$

The constant dilaton is given by $e^\Phi = \frac{L}{k}$ and NS 2-form is gauge trivial. The RR sector, in terms of our notation (5.6), has the following non-zero seven-dimensional components

$$\begin{aligned} G_2 &= -\frac{k}{2} \cos^2 \zeta \sin \theta_1 d\theta_1 \wedge d\phi_1, & J_1^3 &= -k \sin \zeta \cos \zeta d\zeta, & K_0^1 &= -\frac{kL}{2} \sin^2 \zeta, \\ G_4 &= -\frac{3kL^2}{8} \text{vol}(\text{AdS}_4). \end{aligned} \quad (7.17)$$

¹⁵Here we use a specific representation of the gamma matrices in 6d,

$$\begin{aligned} \gamma^\zeta &= 1 \otimes \sigma^2 \otimes \sigma^1, & \gamma^{\theta_1} &= 1 \otimes \sigma^2 \otimes \sigma^3, \\ \gamma^{\phi_1} &= \sigma^1 \otimes 1 \otimes \sigma^2, & \gamma^1 &= \sigma^3 \otimes 1 \otimes \sigma^2, \\ \gamma^2 &= \sigma^2 \otimes \sigma^1 \otimes 1, & \gamma^3 &= \sigma^2 \otimes \sigma^3 \otimes 1, \end{aligned} \quad (7.14)$$

where σ^i are the Pauli matrices.

The $SU(2)$ transformation of this solution was performed in [18], where it was claimed that the generated geometry describes a strongly-coupled three-dimensional quiver preserving $\mathcal{N} = 2$ supersymmetry. It was assumed that the Kosmann derivative suitably captured the preserved supersymmetry and in this paper we confirm that the assumption is correct.

A new AdS_3 solution preserving 16 Supercharges

In this section we generate a new AdS_3 solution in type IIA supergravity by performing an $SU(2)$ transformation on a well-known $AdS_3 \times S^3 \times S^3 \times \mathbb{R}$ geometry. We consider the solution in type IIB presented in [45], which corresponds to the near-horizon limit of two stacks of D5 branes with D1 branes lying at their intersection and smeared elsewhere [57, 58, 59] (see also [60]). The metric may be expressed as

$$ds^2 = L^2 ds^2(AdS_3) + R_1^2 ds^2(S_1^3) + R_2^2 ds^2(S_2^3) + dx^2, \quad (7.18)$$

while the dilaton is constant. The metric is supported by a non-trivial RR three-form

$$F_3 = 2L \text{vol}(AdS_3) + 2R_1 \text{vol}(S_1^3) + 2R_2 \text{vol}(S_2^3). \quad (7.19)$$

From the EOMs we get the constraint on the radii

$$\frac{1}{L^2} = \frac{1}{R_1^2} + \frac{1}{R_2^2}. \quad (7.20)$$

This solution has attracted some recent interest, particularly so since a proposal for the dual CFT was given in [61] (see [45, 62] for earlier work). The dual CFT is a $\mathcal{N} = (4, 4)$ gauge theory with two $SU(2)$ R-symmetries, realised in the geometry by the two round 3-spheres.

Indeed, from a supersymmetry perspective, this example bears some resemblance to the Brandhuber-Oz solution [16], since although we have $SO(4)^2 \times U(1)$ isometry, the R-symmetry is just $SU(2) \times SU(2)$. This should be compared with the warped $AdS_6 \times S^4$ solution [16] where the warping reduces the internal $SO(5)$ isometry to $SO(4)$. The remaining $SO(4)$ can be further decomposed into $SU(2) \times SU(2)$, where only one factor corresponds to the R-symmetry. Therefore, given that the Brandhuber-Oz solution permits a supersymmetry preserving $SU(2)$ T-dual [14], it is expected that we can perform two such transformations here. We leave a proper analysis of this geometry to future work and content ourselves here with discussing the supersymmetry and presenting the T-dual geometry. For simplicity, we will only consider the transformation on one three-sphere, but in principle both can be done. One may also consider the diagonal $SU(2)$ and it is expected that supersymmetry will be broken (for example [63]).

To show that supersymmetry will be preserved, given the simplicity of the initial geometry, it is enough to work out the Killing spinors explicitly. Let us introduce the following vielbeins

$$e^{x^\mu} = L dx_\mu, \quad e^r = L \frac{dr}{r}, \quad e^x = dx, \quad e^{\tilde{a}} = \frac{R_1}{2} \tilde{\tau}_a, \quad e^a = \frac{R_2}{2} \tau_a, \quad (7.21)$$

where $a = 1, 2, 3$ and $\tilde{a} = \tilde{1}, \tilde{2}, \tilde{3}$. The spin connection is particularly simple with

$$\omega^{x^\mu t} = \frac{1}{L} e^{x^\mu}, \quad \omega^{\tilde{a} \tilde{b}} = \frac{1}{R_1} \epsilon_{abc} e^{\tilde{c}}, \quad \omega^{ab} = \frac{1}{R_2} \epsilon_{abc} e^c, \quad (7.22)$$

being the only non zero components. The dilatino condition gives rise to the projection

$$\frac{1}{L}\Gamma^{x^0x^1r}\sigma_1\eta + \frac{1}{R_1}\Gamma^{\bar{1}\bar{2}\bar{3}}\sigma_1\eta + \frac{1}{R_2}\Gamma^{123}\sigma_1\eta = 0, \quad (7.23)$$

which preserves 16 supercharges. It is quick to see that the projection kills all but the AdS_3 components of the gravitino variation leaving the differential condition

$$\nabla_a\eta = \Gamma_a\Gamma^{x^0x^1r}\sigma_1\eta, \quad a = x^0, x^1, r, \quad (7.24)$$

which is recognisable as the standard Killing spinor equation for AdS_3 . Therefore the Killing spinor is of the form

$$\eta = \mathcal{M}(\text{AdS}_3)\eta_0, \quad (7.25)$$

where η_0 is a constant spinor obeying the projection condition in eq (7.23) and \mathcal{M} is a matrix which commutes with the projection. As the spinor of the original solution depends only on the coordinates of AdS_3 we conclude that 16 supercharges are preserved by the new solution.

We will now use our earlier results to T-dualise with respect to one of the $\text{SU}(2)$ isometries of S^3_2 and from here on set $S^3_1 = S^3$. We read off the following input for the $\text{SU}(2)$ rules

$$e^{2C_a} = \frac{R_2^2}{4}, \quad a = 1, 2, 3, \\ G_3 = 2L \text{vol}(\text{AdS}_3) + 2R_1 \text{vol}(S^3), \quad m = \frac{R_2^2}{4}, \quad (7.26)$$

with all other fields zero. We note that the appearance of a non-zero Romans' mass m indicates that the T-dual solution will be a massive type IIA solution. We can immediately read off the T-dual solution, however to make the remnant $\text{SU}(2)$ symmetry manifest, it is advantageous to express the T-dual geometry in spherical polar coordinates

$$v_1 = \rho \sin \chi \cos \xi, \quad v_2 = \rho \sin \chi \sin \xi, \quad v_3 = \rho \cos \chi. \quad (7.27)$$

The resulting metric is

$$ds^2_{IIA} = L^2 ds^2(\text{AdS}_3) + dx^2 + R_1^2 ds^2(S^3) + \frac{4}{R_2^2} \left(d\rho^2 + \frac{R_2^6 \rho^2}{64\Delta} ds^2(S^2) \right), \quad (7.28)$$

where S^2 , corresponding to the $\text{SU}(2)$ R-symmetry, is the two-sphere spanned by χ and ξ and

$$\Delta = \frac{R_2^6 + 16R_2^2 \rho^2}{64}. \quad (7.29)$$

The dual dilaton is given by

$$e^{-2\Phi} = \Delta, \quad (7.30)$$

while the NS two-form is simply

$$B_2 = \frac{R_2^2 \rho^3}{4\Delta} \text{vol}(S^2). \quad (7.31)$$

The RR sector can be read off from eq. (6.2) and is given by

$$m = \frac{R_2^2}{4}, \quad F_2 = mB_2, \quad (7.32)$$

$$F_4 = -\frac{R_2^3}{4LR_1} [L^4 \text{vol}(AdS_3) + R_1^4 \text{vol}(S^3)] \wedge dx - 2\rho [L^2 \text{vol}(AdS_3) - R_1^2 \text{vol}(S^3)] \wedge d\rho.$$

We have investigated the curvature invariants of this solution and found that they are non-singular. This is in line with our expectations, since we dualised on an S^3 of constant radius.

8 Conclusions

For spacetimes with $SO(4)$ isometry it has been demonstrated that left or right acting $SU(2)$ non-Abelian T-duality is a symmetry of the EOMs of type II supergravity and, in addition, that the supersymmetry conditions before and after T-duality are mapped into each other up to the vanishing of the Kosmann spinorial Lie derivative [20]. Since a great number of interesting geometries, for example [10, 18, 19], fall outside the scope of this earlier work, in this paper we have extended the supersymmetry analysis of Ref. [20] to a large class of solutions with Bianchi IX symmetry. In analogy with the Abelian case, we have noted that the Kosmann derivative captures spacetime supersymmetry breaking. Therefore, any question regarding spacetime supersymmetry can be settled by studying the Kosmann derivative, or alternatively, by explicitly working out the Killing spinors of the original geometry. It should be instructive to recast these findings in terms of the language of pure spinors.

It is expected that an underlying inert seven-dimensional theory can be found via consistent Kaluza-Klein reduction in a similar vein to [20]. This would provide an effective, yet formal proof that the $SU(2)$ transformation is also a symmetry of the EOMs for the general Ansatz we consider. Evidence suggests that this theory is not a supergravity and will possess massive modes [20]. Given that supersymmetry under mild assumptions implies certain EOMs, it may be more elegant to provide a proof based on supersymmetry, and we leave this to future work.

At this stage it is firmly believed that non-Abelian T-duality takes supergravity solutions to supergravity solutions. However, there is currently no definitive statement when α' -corrections to supergravity are considered. This motivates the future study of non-Abelian T-duality in the context of Heterotic supergravity, where α' corrections appear at the linear level, to determine whether non-Abelian T-duality is simply a symmetry of the strict supergravity limit, or not. One may eventually consider extending this avenue of enquiry to higher-order corrections, a direction of research that has been pursued in the Abelian case in, for instance, [64, 65].

Indeed, the form of the transformation (5.2) is already deeply suggestive, since in various limits, for example $z_1 \rightarrow \infty$, z_2, z_3 finite, we recover the results of Abelian T-duality. Thus, by treating the original and T-dual coordinates on an equal footing, as is the case in Double Field Theory (DFT) [66, 67], it should be possible to combine Abelian and non-Abelian T-duality. Such a generalisation already exists in the literature and goes under the name Poisson-Lie T-duality [68, 69], where a natural counterpart of $O(d,d)$ symmetry involves automorphisms of the Drinfeld double, the algebraic structure underpinning Poisson-Lie

T-duality. An analogue of DFT for non-Abelian T-duality promises to extend the known class of non-geometric compactifications and preliminary work in this direction is currently underway [47].

Given that non-Abelian T-duality is expected to preserve integrability, another exciting open direction is to consider using it as a means to generate integrable deformations of AdS/CFT geometries. Given backgrounds with $U(1) \times U(1)$ isometry, it is a well-known fact that integrable deformations based on TsT transformations exist [33]. The counterpart in our setting will involve some mixing between residual isometries and a necessary inverse non-Abelian T-duality, in contrast to the usual Abelian case. More generally, there is an open question about how non-Abelian T-duality combines with $SL(2, \mathbb{R})$ (S-duality) transformations in the supergravity and whether this leads to new examples of compact geometries. In this sense, the role of non-Abelian T-duality in solution generating, namely as an intermediary step, may mirror fermionic T-duality [70]. We hope to report on this in future work.

There is also an ongoing program of work where non-Abelian T-dual geometries are being studied in the context of the AdS/CFT correspondence in order to infer properties of putative dual CFTs [9, 10, 18, 19]. Certain examples have been proved to preserve $\mathcal{N} = 1$ in four dimensions by an explicit illustration of the dual G -structures defined on the six-dimensional internal space [22, 24]. Although such an analysis provides additional geometric information about these solutions, the results here are not confined by dimensionality and make any question of supersymmetry rather trivial in these examples. All these solutions are independent of the $SU(2)$ directions in the preferred frame [71], and so supersymmetry is preserved.

There are several novel examples that have yet to be studied, but we can be confident supersymmetry will be preserved under the $SU(2)$ transformation. Examples include the holographic duals of flows from $\mathcal{N} = 4$ super-Yang-Mills [72, 73, 74]. Indeed, non-Abelian T-duality on the $\mathcal{N} = 1$ fixed-point solution [73] provides a natural generalisation of the recent work of Ref. [19] for five-manifolds not in the Sasaki-Einstein class. One might go further and consider non-Abelian T-dualising the known flows between S^5 and $T^{1,1}$ [75, 76] with the hope of learning something about flows between the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Sicilian quivers¹⁶. Owing to the fact that the T-dual geometry and preserved supersymmetry can be simply read off from our results, there are now numerous other possibilities to consider.

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¹⁶These were argued to be related to the non-Abelian T-dual of $T^{1,1}$ and S^5 respectively in [10] and [7].

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A Bianchi IX symmetry

In this paper we focus on internal spaces with $SU(2)$ isometry corresponding to diagonal Bianchi IX symmetry. In terms of left-invariant one-forms, τ_a , these spaces may be written as

$$ds_{10}^2 = ds_7^2 + \sum_{a=1}^3 e^{2C_a} (\tau_a + A^a)^2, \quad (\text{A.1})$$

where C_a and A^a depend on transverse directions. When the scalars C_a are completely generic, the metric has a right-acting $SU(2)$ isometry. When two of the C_a are the same, we find the enhancement to $SU(2) \times U(1)$ isometry and when all C_a are the same, we recover the round S^3 . In the last case, we have $SO(4) \simeq SU(2) \times SU(2)$ isometry and the Killing vectors can be divided into right-invariant

$$\begin{aligned} K_1 &= -\frac{\cos \phi}{\sin \theta} \partial_\psi + \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi, \\ K_2 &= \frac{\sin \phi}{\sin \theta} \partial_\psi + \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi, \\ K_3 &= \partial_\phi, \end{aligned} \quad (\text{A.2})$$

and left-invariant vectors, which are dual to the one-forms appearing in the metric (A.1),

$$\begin{aligned} \tilde{K}_1 &= -\cot \theta \cos \psi \partial_\psi - \sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} \partial_\phi, \\ \tilde{K}_2 &= -\cot \theta \sin \psi \partial_\psi + \cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} \partial_\phi, \\ \tilde{K}_3 &= \partial_\psi. \end{aligned} \quad (\text{A.3})$$

For the metric to be invariant under these symmetries, we require that

$$\mathcal{L}_X g_{\mu\nu} = X^\xi \partial_\xi g_{\mu\nu} + \partial_\mu X^\lambda g_{\lambda\nu} + \partial_\nu X^\lambda g_{\mu\lambda} = 0, \quad (\text{A.4})$$

where X is a vector field. It is easy to check that when the C_a are generic, the isometries are broken to $SU(2)$ corresponding to the right-invariant vectors, K_a .

Requiring that the Ansatz for the fluxes has the same $SU(2)$ isometry with respect to the right-invariant vectors, fixes them to be written in terms of wedge products of the left-invariant one-forms as presented in the body of the text. Again, a short calculation reveals that for $\omega = (\tau_a + A^a)$

$$\mathcal{L}_{K_a} \omega = i_{K_a} d\omega + di_{K_a} \omega = 0, \quad (\text{A.5})$$

where K_a denotes right-invariant vector fields.

We remark that a more general metric with a right-acting $SU(2)$ isometry takes the form

$$ds^2 = \sum_{a,b=1}^3 e^{2C_{ab}} (\tau_a + A^a) \otimes (\tau_b + A^b). \quad (\text{A.6})$$

While one can always diagonalise the metric at a given point of the transverse space, in general the metric has off-diagonal components.

B Technical Details

In this section we house information about the T-dual field strength \hat{H}_3 for the NS two-form and the T-dual spin connection, $\hat{\omega}$, both of which appear in the T-dual Killing spinor equations.

In our chosen frame (4.16), the field strength, $\hat{H}_3 = d\hat{B}_2$, may then be written as

$$\begin{aligned}
\hat{H}_3 = & -e^{-C_1-C_2-C_3} \left[\sum_a e^{2C_a} - \frac{2}{\Delta} (\sum_a e^{4C_a} z_a^2) \right] \hat{e}^{123} \\
& + \frac{2}{\Delta} e^{C_1+C_3} [dC_1(e^{2C_2} z_2 - z_1 z_3) + dC_3(e^{2C_2} z_2 + z_1 z_3)] \hat{e}^{31} \\
& + \frac{2}{\Delta} e^{C_1+C_2} [dC_2(e^{2C_3} z_3 - z_1 z_2) + dC_1(e^{2C_3} z_3 + z_1 z_2)] \hat{e}^{12} \\
& + \frac{2}{\Delta} e^{C_2+C_3} [dC_3(e^{2C_1} z_1 - z_2 z_3) + dC_2(e^{2C_1} z_1 + z_2 z_3)] \hat{e}^{23} \\
& - \frac{1}{\Delta} \left[A^1 e^{C_1} (z_2 e^{C_2} + z_1 z_3 e^{-C_2}) (e^{2C_2} - e^{2C_3}) + A^2 e^{C_2} (e^{C_1} z_1 - z_2 z_3 e^{-C_1}) (e^{2C_3} - e^{2C_1}) \right. \\
& - A^3 e^{-C_1-C_2} (e^{2C_2} z_2^2 - e^{2C_1} z_1^2) (e^{2C_1} - e^{2C_2}) \left. \right] \hat{e}^{12} \\
& - \frac{1}{\Delta} \left[A^2 e^{C_2} (z_3 e^{C_3} + z_1 z_2 e^{-C_3}) (e^{2C_3} - e^{2C_1}) + A^3 e^{C_3} (e^{C_2} z_2 - z_1 z_3 e^{-C_2}) (e^{2C_1} - e^{2C_2}) \right. \\
& - A^1 e^{-C_2-C_3} (e^{2C_3} z_3^2 - e^{2C_2} z_2^2) (e^{2C_2} - e^{2C_3}) \left. \right] \hat{e}^{23} \\
& - \frac{1}{\Delta} \left[A^3 e^{C_3} (z_1 e^{C_1} + z_2 z_3 e^{-C_1}) (e^{2C_1} - e^{2C_2}) + A^1 e^{C_1} (e^{C_3} z_3 - z_1 z_2 e^{-C_3}) (e^{2C_2} - e^{2C_3}) \right. \\
& - A^2 e^{-C_3-C_1} (e^{2C_1} z_1^2 - e^{2C_3} z_3^2) (e^{2C_3} - e^{2C_1}) \left. \right] \hat{e}^{31} \\
& + \frac{1}{\Delta} \left[(e^{2C_2+2C_3} + z_1^2) e^{C_1} \hat{e}^1 + (e^{2C_3} z_3 + z_1 z_2) e^{C_2} \hat{e}^2 - (e^{2C_2} z_2 - z_1 z_3) e^{C_3} \hat{e}^3 \right] e^{2C_1} F^1 \\
& + \frac{1}{\Delta} \left[(e^{2C_3+2C_1} + z_2^2) e^{C_2} \hat{e}^2 + (e^{2C_1} z_1 + z_2 z_3) e^{C_3} \hat{e}^3 - (e^{2C_3} z_3 - z_1 z_2) e^{C_1} \hat{e}^1 \right] e^{2C_2} F^2 \\
& + \frac{1}{\Delta} \left[(e^{2C_1+2C_2} + z_3^2) e^{C_3} \hat{e}^3 + (e^{2C_2} z_2 + z_1 z_3) e^{C_1} \hat{e}^1 - (e^{2C_1} z_1 - z_2 z_3) e^{C_2} \hat{e}^2 \right] e^{2C_3} F^3
\end{aligned}$$

$$\begin{aligned}
& + H \\
& + \frac{1}{\Delta} e^{2C_1} B^1 \left[(e^{2C_2+2C_3} - z_1^2) e^{-C_2-C_3} \hat{e}^{23} + (-z_1 z_2 + z_3 e^{2C_3}) e^{-C_1-C_3} \hat{e}^{31} \right. \\
& \quad \left. - (z_2 e^{2C_2} + z_1 z_3) e^{-C_1-C_2} \hat{e}^{12} \right] \\
& + \frac{1}{\Delta} e^{2C_2} B^2 \left[(e^{2C_3+2C_1} - z_2^2) e^{-C_1-C_3} \hat{e}^{31} + (-z_2 z_3 + z_1 e^{2C_1}) e^{-C_1-C_2} \hat{e}^{12} \right. \\
& \quad \left. - (z_3 e^{2C_3} + z_1 z_2) e^{-C_2-C_3} \hat{e}^{23} \right] \\
& + \frac{1}{\Delta} e^{2C_3} B^3 \left[(e^{2C_1+2C_2} - z_3^2) e^{-C_1-C_2} \hat{e}^{12} + (-z_1 z_3 + z_2 e^{2C_2}) e^{-C_2-C_3} \hat{e}^{23} \right. \\
& \quad \left. - (z_1 e^{2C_1} + z_2 z_3) e^{-C_2-C_3} \hat{e}^{31} \right] \\
& - \frac{1}{\Delta} \left[-(z_2^2 e^{2C_2} + z_3^2 e^{2C_3}) e^{-C_1} \hat{e}^1 + (z_1 z_2 + z_3 e^{2C_3}) e^{C_2} \hat{e}^2 + (z_1 z_3 - z_2 e^{2C_2}) e^{C_3} \hat{e}^3 \right] DB^1 \\
& - \frac{1}{\Delta} \left[-(z_1^2 e^{2C_1} + z_3^2 e^{2C_3}) e^{-C_2} \hat{e}^2 + (z_2 z_3 + z_1 e^{2C_1}) e^{C_3} \hat{e}^3 + (z_1 z_2 - z_3 e^{2C_3}) e^{C_1} \hat{e}^1 \right] DB^2, \\
& - \frac{1}{\Delta} \left[-(z_2^2 e^{2C_2} + z_1^2 e^{2C_1}) e^{-C_3} \hat{e}^3 + (z_1 z_3 + z_2 e^{2C_2}) e^{C_1} \hat{e}^1 + (z_2 z_3 - z_1 e^{2C_1}) e^{C_2} \hat{e}^2 \right] DB^3.
\end{aligned}$$

Here we have omitted wedge products for conciseness and the explicit form of H can be found in (4.8).

In our chosen frame, it is also helpful to record the derivative of the vielbein:

$$\begin{aligned}
d\hat{e}^1 &= -dC_1 \hat{e}^1 + \frac{2}{\Delta} \left[(e^{2C_2} z_2^2 + e^{2C_3} z_3^2) dC_1 \hat{e}^1 + e^{C_1+C_2} (e^{2C_3} z_3 - z_1 z_2) dC_2 \hat{e}^2 \right. \\
& \quad \left. - e^{C_1+C_3} (e^{2C_2} z_2 + z_1 z_3) dC_3 \hat{e}^3 \right] \\
& + \frac{1}{\Delta} \left[A^2 (e^{2C_1} - e^{2C_3}) (e^{2C_2} z_2 + z_1 z_3) + A^3 (e^{2C_1} - e^{2C_2}) (e^{2C_3} z_3 - z_1 z_2) \right] \hat{e}^1 \\
& + \frac{1}{\Delta} \left[-A^1 e^{C_1} (e^{2C_2} - e^{2C_3}) (e^{C_2} z_2 + e^{-C_2} z_1 z_3) - A^3 e^{C_1-C_2} (z_1^2 + e^{2C_2+2C_3}) (e^{2C_1} - e^{2C_2}) \right] \hat{e}^2 \\
& + \frac{1}{\Delta} \left[A^1 e^{C_1} (e^{2C_2} - e^{2C_3}) (e^{C_3} z_3 - e^{-C_3} z_1 z_2) + A^2 e^{C_1-C_3} (e^{2C_2+2C_3} + z_1^2) (e^{2C_1} - e^{2C_3}) \right] \hat{e}^3 \\
& - e^{C_1-C_3} A^2 e^3 + e^{C_1-C_2} A^3 \hat{e}^2 + \frac{e^{C_2}}{\Delta} (e^{C_2} z_2 + z_1 z_3) [1 + (e^{2C_1} - e^{2C_3}) e^{-2C_2}] \hat{e}^{12} \\
& - \frac{e^{C_3}}{\Delta} (e^{2C_3} z_3 - z_1 z_2) [1 + (e^{2C_1} - e^{2C_2}) e^{-2C_3}] \hat{e}^{31} \\
& + e^{C_1-C_2-C_3} [1 + \frac{1}{\Delta} (e^{2C_2} + e^{2C_3} - e^{2C_1}) (z_1^2 + e^{2C_2+2C_3})] \hat{e}^{23}
\end{aligned}$$

$$\begin{aligned}
& + \frac{e^{C_1}}{\Delta} \left[(e^{2C_3} z_3^2 + e^{2C_2} z_2^2) F^1 + e^{2C_2} (e^{2C_3} z_3 - z_1 z_2) F^2 - e^{2C_3} (e^{2C_2} z_2 + z_1 z_3) F^3 \right] \\
& + \frac{e^{C_1}}{\Delta} \left[(e^{2C_2+2C_3} + z_1^2) D B^1 + (-z_3 e^{2C_3} + z_1 z_2) D B^2 + (z_2 e^{2C_2} + z_1 z_3) D B^3 \right] \\
& + \frac{e^{C_1}}{\Delta} \left[B^1 \{ -(z_2 e^{2C_2} + z_1 z_3) e^{-C_2} \hat{e}^2 + (-z_3 e^{2C_3} + z_1 z_2) e^{-C_3} \hat{e}^3 \} \right. \\
& + B^2 \{ (z_2 e^{2C_2} + z_1 z_3) e^{-C_1} \hat{e}^1 - (z_1^2 + e^{2C_2+2C_3}) e^{-C_3} \hat{e}^3 \} \\
& \left. + B^3 \{ (z_3 e^{2C_3} - z_1 z_2) e^{-C_1} \hat{e}^1 + (z_1^2 + e^{2C_2+2C_3}) e^{-C_2} \hat{e}^2 \} \right]. \tag{B.1}
\end{aligned}$$

From here the spin-connection, ω , can be worked out. First one reads off c_{abc} from $de^a = -c_{abc} e^b \wedge e^c$, with the components of ω , following from

$$\omega_{abc} = \frac{1}{2} (c_{abc} + c_{bac} - c_{cab}). \tag{B.2}$$

In determining c_{abc} one should use the fact that expressions are cyclic under $1 \rightarrow 2 \rightarrow 3 \rightarrow 1$.

We record some additional handy identities:

$$\begin{aligned}
d(Dz_1) &= DB^1 - Dz_2 A^3 - z_2 F^3 + Dz_3 A^2 + z_3 F^2, \\
\sum_a z_a D z_a &= \sum_a z_a e^{C_a} \hat{e}^a, \\
Dz_1 Dz_2 Dz_3 &= \Delta e^{-C_1-C_2-C_3} \hat{e}^{123}, \\
d(z_1 Dz_2 Dz_3) &= Dz_1 Dz_2 Dz_3 + (z_2 A^3 - z_3 A^2) Dz_2 Dz_3 - z_1 Dz_1 (Dz_2 A^2 + Dz_3 A^3) \\
&+ z_1^2 (Dz_3 F^3 + Dz_2 F^2) - z_1 (z_2 Dz_2 + z_3 Dz_3) F^1 \\
&- B^1 Dz_2 Dz_3 + z_1 (DB^2 Dz_3 - DB^3 Dz_2), \\
\frac{1}{2} \epsilon_{abc} e^{2C_a} z_a D z_b D z_c &= \Delta \frac{1}{2} \epsilon_{abc} z_a e^{-C_b-C_c} \hat{e}^{bc}, \tag{B.3}
\end{aligned}$$

where other expressions follow by exploiting cyclic symmetry in 1, 2, 3.

C Bianchi

In the body of the paper we have used the gravitino variation along the external seven-dimensional spacetime, $\delta\Psi_\mu$, to match the RR fields before and after T-duality. We noted that the T-dual RR flux (5.7) is then consistent with the remaining supersymmetry conditions. As a further cross-check, it is prudent to check that the Bianchi identities before T-duality are mapped to the Bianchi identities post T-duality.

With the original IIA RR flux Ansatz (5.6), the Bianchi identities turn out to be

$$J_1^a = m B^a + D K_0^a, \tag{C.1}$$

$$dG_2 = m H + J_1^a \wedge F^a, \tag{C.2}$$

$$dN_1 = K_0^a D B^a + J_1^a \wedge B^a = d(K_0^a B^a), \tag{C.3}$$

$$dG_4 = H \wedge G_2 + L_3^a \wedge F^a, \tag{C.4}$$

$$L_3^a = D M_2^a - N_1 \wedge F^a + B^a \wedge G_2 - K_0^a H + \epsilon_{bc}^a D B^b \wedge J_1^c, \tag{C.5}$$

where repeated indices are summed.

The upper two equations follow from $dF_2 = mH_3$, while the lower three equations are the result of $dF_4 = H_3 \wedge F_2$. There are two additional equations, but they are implied:

$$DJ_1^a = mDB^a - \epsilon^a_{bc} K_0^b F^c, \quad (C.6)$$

$$DL_3^a = \epsilon^a_{bc} F^b \wedge M_2^c + H \wedge J_1^a + DB^a \wedge G_2. \quad (C.7)$$

In addition, it is useful to record the following equation of motion:

$$d(e^{\sum_a C_a} *_7 G_4) = -H \wedge N_1 - DB^a \wedge M_2^a + B^a \wedge L_3^a. \quad (C.8)$$

We have checked that the Bianchi identities after T-duality are satisfied. The simplest example by far involves $d\hat{F}_1 = 0$, which can be shown to hold by combining (C.1) and (C.3). This provides an extra check on the $SU(2)$ transformation of the RR sector presented in section 5.

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